TRANSPORT PROCESSES IN THE MIXING OF INHOMOGENEOUS GAS STREAMS

A. L. Mosse and I. T. El'perin

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The equation of the laminar boundary layer is used to analyze the mixing of two plane oncoming (or parallel) streams of inhomogeneous compressible gases.

The problem of the mixing of two homogeneous parallel or oncoming streams of a compressible gas or of a viscous incompressible liquid, under conditions of laminar and turbulent flow, have been investigated in sufficient detail [1–3]. However, in practice we have to deal in the main with mixing processes of oncoming or parallel inhomogeneous streams, as well as with two-phase systems. Turbulent mixing of coaxial streams of the same temperature was considered in [4, 5]. Mixing processes in parallel and oncoming streams of inhomogeneous compressible gases are of considerable practical importance in various technological installations (furnaces, plasma-chemical reactors, etc.).

Below we present the results of a theoretical investigation of transport processes in the mixing of two plane oncoming streams of inhomogeneous compressible gases. We assume the mixing to be a result of molecular diffusion in the boundary layer (the mixing zone) at the interface of the streams, and that the gases being mixed do not chemically interact. All assumptions from boundary-layer theory are applicable, and, furthermore, $\mathrm{Pr} \neq 1 = \mathrm{const}$, $\mathrm{Sc} \neq 1 = \mathrm{const}$, and $\mathrm{Pr} \neq \mathrm{Sc}$.

If thermal diffusion is neglected, the system of differential equations for the plane stationary motion of inhomogeneous compressible gases in the laminar boundary layer (the mixing zone) of two plane streams, after transformation into a dimensionless form normalized with respect to parameters of the unperturbed flow in the region $y > 0$, will be of the form

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{2}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{\mathrm{Pr}} \frac{\partial h}{\partial y}\right) + (k-1)\mathrm{M}_1^2 \frac{\partial}{\partial y}\left[\left(1-\frac{1}{\mathrm{Pr}}\right)\mu \frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right], \tag{3}$$

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{\mathrm{Sc}} \frac{\partial C_i}{\partial y}\right), \tag{4}$$

$$\rho h = 1; \ \mu = h^n \tag{5}$$

with boundary conditions in the region $x > 0$ (mixing of streams begins at $x = 0$)

$$\left.\begin{aligned} u = 1;\ h = 1;\ C_i = 1 \ \text{for}\ y = +\infty \\ u = m_u;\ h = h_m;\ C_i = m_c \ \text{for}\ y = -\infty \end{aligned}\right\}, \tag{6}$$

where

$$m_u = \frac{u_2}{u_1};\ h_m = \frac{h_2}{h_1};\ m_c = \frac{C_2}{C_1}.$$

In the case of parallel streams velocity u_2 is positive, while for oncoming streams it is negative; u_1 is always greater than 0 ($m_u = 0$ relates to the problem of the stream boundary).

For the time being, let us assume a linear dependence of the viscosity coefficient on enthalpy; we have $n = 1$, and $\mu = h$ (we must, however, bear in mind that this is valid in the region of comparatively low temperatures); we use the Dorodnitsyn variables $\xi = x$; $\eta = \int_0^y \rho\, dy$ for the transformation of Eqs. (1)–(4). Let us further assume that for Pr = const and Sc = const the longitudinal velocity component, the enthalpy, and the concentration are functions of only one variable

$$u(\xi, \eta) = \varphi'(\zeta);$$
$$h(\xi, \eta) = h(\zeta);\ C_i(\xi, \eta) = C_i(\zeta);\ \zeta = \eta/2\sqrt{\xi};$$

thus we finally obtain the system of independent ordinary differential equations

$$\varphi''' + 2\varphi\varphi'' = 0, \tag{7}$$

$$h'' + 2\mathrm{Pr}\,\varphi h' + \frac{1}{2}(k-1)\mathrm{M}_1^2(\mathrm{Pr}-1)(\varphi''')^2 = 0, \tag{8}$$

$$C_i'' + 2\mathrm{Sc}\, C_i' \varphi = 0 \tag{9}$$

with boundary conditions

$$\left.\begin{aligned} \varphi' = 1;\ h = 1;\ C_i = 1 \ \text{for}\ \zeta = +\infty \\ \varphi' = m_u;\ h = h_m;\ C_i = m_c \ \text{for}\ \zeta = -\infty \end{aligned}\right\}. \tag{10}$$

Equation (7) with its related boundary conditions is analogous to the equation of the dynamic problem involved in the mixing of homogeneous compressible gas streams and its boundary conditions—whose solution is given in [2]—and the results are valid in this case as well.

The solution of the energy equation (8) will be obtained by the method of parameter variation.

Using the substitution from Eq. (7)

$$\varphi = -\frac{\varphi'''}{2\varphi''};$$

$$\int_{-\infty}^{\zeta} 2\varphi\, d\eta = -\int_{-\infty}^{\zeta} \frac{\varphi'''}{\varphi''}\, d\eta = -\ln\frac{\varphi''(\zeta)}{\varphi''(0)}, \tag{11}$$

taking the value of the function $\varphi''(\zeta)$ from [3], and introducing the notation

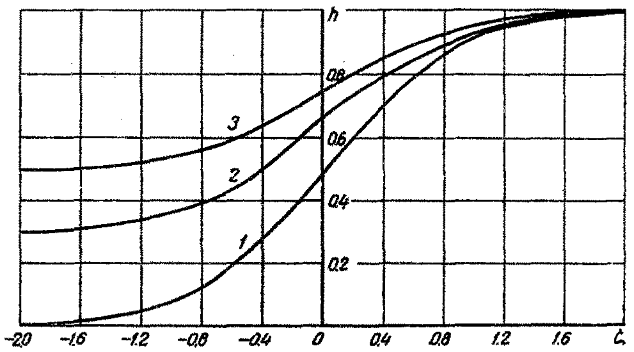

Fig. 1. Profiles of total enthalpy in the mixing zone of two inhomogeneous streams at Pr = 1: 1) $h_m = 0$; 2) 0.3; 3) 0.5.

$$R(-\infty,\ \infty) = \int_{-\infty}^{\infty} [\varphi''(\zeta)]^{\mathrm{Pr}} \int_{-\infty}^{\zeta} [\varphi'''(\zeta)]^2 [\varphi''(\zeta)]^{-\mathrm{Pr}}\, d\zeta\, d\zeta, \qquad (12)$$

$$R(-\infty,\ \zeta) = \int_{-\infty}^{\zeta} [\varphi''(\zeta)]^{\mathrm{Pr}} \int_{-\infty}^{\zeta} [\varphi'''(\zeta)]^2 [\varphi''(\zeta)]^{-\mathrm{Pr}}\, d\zeta\, d\xi, \qquad (13)$$

we finally obtain the general solution of Eq. (13) in the form

$$h = h_m + \frac{1}{2}(1 - h_m)(1 + \operatorname{erf}\zeta\sqrt{\mathrm{Pr}}) + \frac{1}{2}(k-1)\mathrm{M}_1^2(\mathrm{Pr}-1)\left[\frac{1}{2}(1 + \operatorname{erf}\zeta\sqrt{\mathrm{Pr}}) \times R(-\infty,\ \infty) - R(-\infty,\ \zeta)\right]. \qquad (14)$$

Relationship (14) is considerably simplified for Pr = 1. We then have

$$h = h_m + \frac{1}{2}(1 - h_m)(1 + \operatorname{erf}\zeta). \qquad (15)$$

When Pr ≠ 1 = const it is necessary to integrate functions $R(-\infty, \infty)$ and $R(-\infty, \zeta)$. Using the value of the function $\varphi''(\zeta)$, we integrate the integrand of the second integral in (12) and (13). As a result we obtain

$$\int_{-\infty}^{\zeta} [\varphi'''(\zeta)]^2 [\varphi''(\zeta)]^{-\mathrm{Pr}}\, d\zeta = \frac{4(1-m_u)^2}{\pi} \int_{-\infty}^{\zeta} \zeta^2 \exp[-(2-\mathrm{Pr})\zeta^2]\, d\zeta =$$

$$= \frac{(1-m_u)^2}{(2-\mathrm{Pr})\pi} \left\{ \frac{\sqrt{\pi}}{(2-\mathrm{Pr})^{1/2}} - 2\zeta \exp[-(2-\mathrm{Pr})\zeta^2] - \frac{\sqrt{\pi}}{(2-\mathrm{Pr})^{1/2}} \operatorname{erf}[(2-\mathrm{Pr})^{1/2},\ \zeta] \right\}. \qquad (16)$$

Then, in accordance with [6],

$$R(-\infty,\ \infty) = \frac{(1-m_u)^2}{\pi\sqrt{\mathrm{Pr}}(2-\mathrm{Pr})^{3/2}} [\pi + (2-\mathrm{Pr})^{-1/2}], \qquad (17)$$

$$R(-\infty,\ \zeta) = \frac{(1-m_u)^2}{\pi(2-\mathrm{Pr})} \times \left\{ \frac{\sqrt{\pi}}{(2-\mathrm{Pr})^{1/2}} \int_{-\infty}^{\zeta} \exp[-\zeta^2 \mathrm{Pr}]\, d\zeta - 2\int_{-\infty}^{\zeta} \zeta \exp[-2\zeta^2]\, d\zeta - \frac{\sqrt{\pi}}{(2-\mathrm{Pr})^{1/2}} \times \right.$$

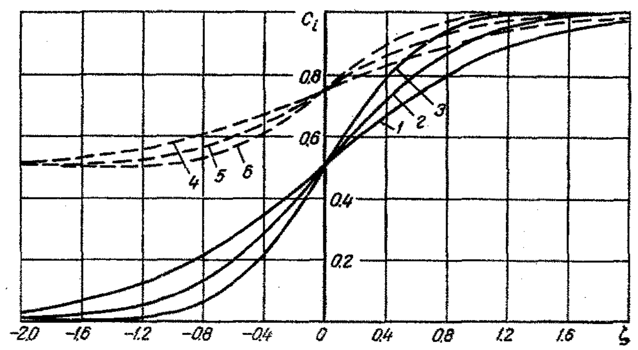

Fig. 2. Profiles of concentration C_i in the mixing zone of two inhomogeneous streams as functions of the generalized parameter $\zeta = \eta/2\sqrt{\xi}$: 1) Sc = 0.5; 2) 1.0; 3) 2.0 (for $m_c = 0$); 4) Sc = 0.5; 5) 1.0; 6) 2.0) for $m_c = 0.5$).

$$\times \int_{-\infty}^{\zeta} \operatorname{erf}[(2-\mathrm{Pr})^{1/2}, \zeta] \exp[-\zeta^2 \mathrm{Pr}]\, d\zeta. \qquad (18)$$

Let us now consider the diffusion problem. From the solution of Eq. (9) with boundary conditions (10) and substitution (11) taken into account, after double integration we obtain a solution in the form

$$C_i(\zeta) = \Big[m_c \int_{\zeta}^{\infty} (\varphi'')^{\mathrm{Sc}}\, d\zeta +$$

$$+ \int_{-\infty}^{\zeta} (\varphi'')^{\mathrm{Sc}}\, d\zeta\Big] \Big[\int_{-\infty}^{\infty} (\varphi'')^{\mathrm{Sc}}\, d\zeta\Big]^{-1} \qquad (19)$$

or, substituting the expression for the function $\varphi''(\zeta)$, after transformation, we obtain

$$C_i(\zeta) = \frac{1}{2}\,[(m_c+1)-(m_c-1)\operatorname{erf}\zeta\sqrt{\mathrm{Sc}}]. \qquad (20)$$

For Sc = 1

$$C_i(\zeta) = \frac{1}{2}\,[(m_c+1)-(m_c-1)\operatorname{erf}\zeta]. \qquad (21)$$

The nature of the gas motion in the mixing zone is analyzed in detail in [1–3].

We now consider the enthalpy and concentration distribution in the mixing region of two oncoming (or parallel) streams of inhomogeneous compressible gases on the basis of obtained results (14), (15), (20), and (21). The results of calculating the dependence of the profile of the total enthalpy h on the generalized parameter $\zeta = \eta/2\sqrt{\xi}$ for various values of h_m and Pr = 1 is shown in Fig. 1. The effective width of the mixing zone obviously increases with an increasing parameter h_m. The dependence of concentration in the mixing region on this parameter for various values of Sc and m_c is shown in Fig. 2. The nature of this dependence indicates that the effective thickness of the diffusion boundary layer increases with decreasing number S_c, and vice versa. The effect of parameter m_c on the variation of the effective width of the mixing zone is, however, more pronounced, particularly in the region $\zeta < 0$.

The final solution of this problem and the feasibility of practical calculations of enthalpy and concentration distribution in the mixing zone of two oncoming streams, as well as the determination of the effect of the stream parameters on the nature of the distribution, necessitate the transition from the ξ, η coordinates to the physical plane coordinates x, y. As is usual in heat problems, this is carried out with the formulas

$$\xi = x;\quad \eta = \int_0^y \frac{\rho}{\rho_\infty}\, dy = \int_0^y \frac{h_1}{h}\, dy. \qquad (22)$$

It follows from (22) that the generalized parameter $\zeta = \eta/2\sqrt{x}$ is related to coordinates x, y by the formula

$$\zeta = \frac{1}{2}\sqrt{\frac{u_1\rho_1}{\mu_1 x}} \int_0^y \frac{h_1}{h}\, dy. \qquad (23)$$

The specific form of (23) depends on the enthalpy distribution. Differentiating with respect to y, we obtain

$$\frac{d\zeta}{dy} = \frac{1}{2}\sqrt{\frac{u_1\rho_1}{\mu_1 x}}\, \frac{h_1}{h(\zeta)}, \qquad (24)$$

from which the relationship between ζ and $y/x^{1/2}$ is determined by the following integral equation

$$\frac{y}{2\sqrt{x}} = \frac{1}{h_1}\sqrt{\frac{\mu_1}{u_1\rho_1}} \int_0^{\zeta} h(\zeta)\, d\zeta. \qquad (25)$$

Since the mass concentration of the i-th component is $C_i = \rho_i/\rho$, the transformation formula for η in the case of the diffusion problem will be

$$\eta = \int_0^y \frac{1}{C_i}\, dy. \qquad (26)$$

After transformations similar to (23) and (24), we obtain

$$\frac{y}{2\sqrt{x}} = \sqrt{\frac{\mu_1}{u_1\rho_1}} \int_0^{\zeta} C_i(\zeta)\, d\zeta. \qquad (27)$$

Certain results of calculations of the dependence of enthalpy h on the dimensionless coordinate $y(u_1\rho_1/\mu_1 x)^{1/2}$ for Pr(Sc) = 1, and various $h_m(m_c)$ are presented in Fig. 3. As in the case of dependence $h = f(\zeta)$ (see Fig.

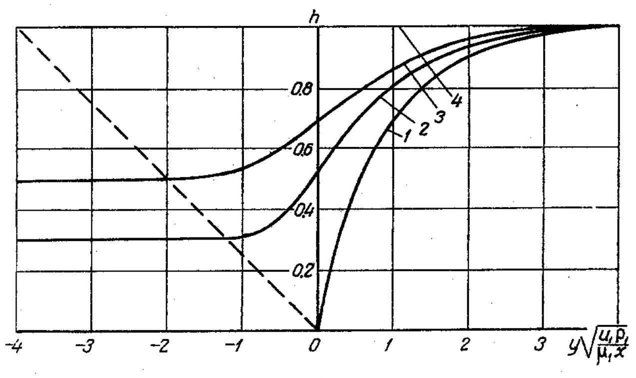

Fig. 3. The dependence of $h(C_1)$ on the dimensionless coordinate $y(u_1\rho_1/\mu_1 x)^{1/2}$ for Pr(Sc) = 1: 1) $h_m(m_c) = 0$; 2) 0.3; 3) 0.5; 4) 1.0.

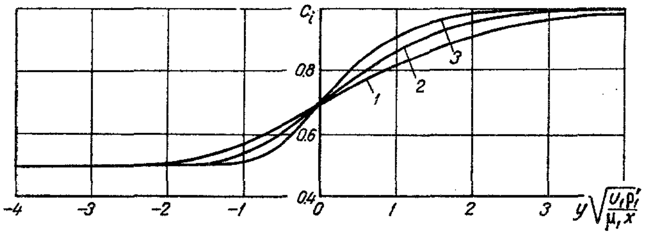

Fig. 4. The dependence of C_i on the dimensionless coordinate $y(u_1\rho_1/\mu_1 x)^{1/2}$ for $m_c = 0.5$: 1) Sc = 0.5; 2) 1.0; 3) 2.0.

1), the width of the mixing zone noticeably decreases with decreasing h_m, which may be explained by the convergence of enthalpies h_1 and h_2 of the streams as $h_m \to 0$. In this case the equalization of the enthalpies, under otherwise equal conditions, occurs much faster, and the mixing zone is smaller. The effect of Sc on function $C_i = f(y(u_1\rho_1/\mu_1 x)^{1/2})$ is shown in Fig. 4 for the parameter $m_c = 0.5$. Clearly, the variation of Sc within the limits of 0.5–2.0 results in lesser variation of the mixing zone dimensions than the variation of m_c from 0 to 1.0.

NOTATION

x and y are, respectively, the coordinates along and normal to the stream axis; u and v are the velocity components along the x- and y-axes, respectively; ρ is the gas density; μ is the dynamic viscosity coefficient; p is the pressure; h is the enthalpy; k is the ratio of specific heats at constant pressure and constant volume; C_i and m_i are, respectively, the concentration by weight, and the molecular weight of the i-th component; M_1 is the Mach number; and $Pr = \mu c_p/\lambda$, and $Sc = \mu/\rho D_i$ are, respectively, the Prandtl and the Schmidt numbers.

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Institute of Heat and Mass Transfer AS BSSR, Minsk